Multi-Soliton-Like Solutions of a Coupled Kadomtsev-Petviashvili System

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Arising in the context of random matrix theory, the coupled Kadomtsev-Petviashvili (KP) systems have been a subject of active studies. In this paper, a coupled KP system with three potentials is investigated with symbolic computation, and the Darboux transformations of its reduced equations are obtained. Moreover, the multi-soliton-like solutions of the coupled KP system are derived. Those solutions could be of some value for the studies in the context of random matrix theory.

Key words: Coupled Kadomtsev-Petviashvili System; Darboux Transformation; Soliton-Like Solution; Symbolic Computation.

1. Introduction

Due to the limitation of the dimension, the pure one-dimensional systems can not account for some observed features [1]. In realistic situations, the higher dimensional systems may provide more useful models [1, 2]. As a typical example in (2+1)-dimensional systems, the Kadomtsev-Petviashvili (KP) system has been derived from many physical applications in plasma physics, fluid dynamics, water waves, astrophysics, cosmology, optics, and Bose-Einstein condensation [1, 3]. Compared with the KP equation, the coupled one possesses soliton solutions with more parametric freedom [4]. Therefore, its solutions can be expected to model more complex situations in reality than the KP equation [5, 6]. Recent research has claimed that the coupled KP system has an inner connection with matrix integrals over the orthogonal and symplectic ensembles, so the solutions might be applied to the context of random matrix theory [5, 7]. The coupled KP system has been proposed as the soliton system through coupling the KP system to the Davey-Stewartson one [6, 8 – 10]. Since then, the coupled KP system has been reconstructed several times from different points of view [7, 11].

In this paper, with symbolic computation [12 – 14], we will investigate a coupled KP system with three potentials in the following form [15]:

$$
\begin{aligned}
q_t &= \frac{1}{4}\left(q_{xxx} - 6qq_x + 3\int q_{yy}\mathrm{d}x + 6(pr)_x\right),\\
p_t &= \frac{1}{2}\left(-p_{xxx} + 3qp_x + 3p\int q_y\mathrm{d}x - 3p_{xy}\right),\\
r_t &= \frac{1}{2}\left(-r_{xxx} + 3qr_x - 3r\int q_y\mathrm{d}x + 3r_{xy}\right),
\end{aligned}
\tag{1}
$$

where the subscripts represent the partial derivatives. Through certain reductions [11], System (1) can be reduced to the coupled Korteweg-de Vries and standard KP equations [4, 5].

In recent years, the coupled KP systems have received certain interest [5, 6, 9, 11, 15 – 17]. The Lax pair and algebraic-geometrical solutions in terms of the Riemann theta functions of System (1) have been given [15]. An elementary Bäcklund-Darboux transformation has been constructed by the spin-representation formulation [16]. Some typical and spider-web solutions of System (1) have been presented [5, 6].

Author of [15] has decomposed System (1) into the first two members of the (1+1)-dimensional Ablowitz-Kaup-Newell-Segur hierarchy,

$$u_y = -u_{xx} + 2u^2v, \quad v_y = v_{xx} - 2uv^2, \tag{2}$$

$$u_t = u_{xxx} - 6uvu_x, \quad v_t = v_{xxx} - 6uvv_x, \tag{3}$$

and obtained the following proposition:

If (u,v) is a compatible solution of (2) and (3), then the function (p,q,r) determined by

$$q = 4uv, \quad p = -2u^2, \quad r = -2v^2, \tag{4}$$

is a solution of System (1). It should be noted that this restricted class of solutions is for System (1), and the KP-II equation as well, under the restriction of $q^2 = 4pr$, i. e. (4), the first equation in (1) is just the KP-II equation.

Authors of [5, 6, 11] have constructed the multi-soliton solutions of System (1) in terms of the functional forms and Pfaffians. Based on the proposition of [15], this paper will derive the multi-soliton solutions of System (1) by use of the Darboux transformation (DT). Those results might be useful in the context of random matrix theory.

The outline of this paper will be as follows. In Section 2, we will present three kinds of DTs for (2) and (3). In Section 3, based on the obtained three DTs and (4), the one-, two-, and three-soliton-like solutions of System (1) will be presented. Section 4 will be our conclusions.

2. Darboux Transformations for (2) and (3) with Symbolic Computation

The DT which can be applied to a class of nonlinear evolution equations is a computerizable procedure [18 – 19], and the DT can give rise to a procedure to recursively generate a series of analytic solutions including the multi-soliton solutions from an initial solution [20, 21].

In this section, we will present three kinds of DTs for (2) and (3). The Lax representation of (2) and (3) has the following form [15]:

$$\boldsymbol{\Psi}_x = U\boldsymbol{\Psi}, \quad \boldsymbol{\Psi}_y = V\boldsymbol{\Psi}, \quad \boldsymbol{\Psi}_t = W\boldsymbol{\Psi} \tag{5}$$

with

$$U = \begin{pmatrix} -\frac{1}{2}\lambda & u \\ v & \frac{1}{2}\lambda \end{pmatrix}, \tag{6}$$

$$V = \begin{pmatrix} -\frac{1}{2}\lambda^2 + uv & u\lambda - u_x \\ v\lambda + v_x & \frac{1}{2}\lambda^2 - uv \end{pmatrix}, \tag{7}$$

$$W = \tag{8}$$
$$\begin{pmatrix} -\frac{1}{2}\lambda^3 + uv\lambda - vu_x + uv_x & u\lambda^2 - u_x\lambda - 2u^2v + u_{xx} \\ v\lambda^2 + v_x\lambda - 2uv^2 + v_{xx} & \frac{1}{2}\lambda^3 - uv\lambda + vu_x - uv_x \end{pmatrix},$$

where $\boldsymbol{\Psi} = (\psi_1(x,y,t), \psi_2(x,y,t))^{\mathrm{T}}$, T denotes the transpose of the vector, and λ is the isospectral parameter. From the compatibility conditions $U_y - V_x + [U,V] = 0$ and $U_t - W_x + [U,W] = 0$, (2) and (3) can be derived.

We can take the DT as the form,

$$\hat{\boldsymbol{\Psi}} = D\boldsymbol{\Psi}, \tag{9}$$

which satisfies

$$\hat{\boldsymbol{\Psi}}_x = \hat{U}\hat{\boldsymbol{\Psi}}, \quad \hat{\boldsymbol{\Psi}}_y = \hat{V}\hat{\boldsymbol{\Psi}}, \quad \hat{\boldsymbol{\Psi}}_t = \hat{W}\hat{\boldsymbol{\Psi}}, \tag{10}$$

and

$$D_x + DU - \hat{U}D = 0, \tag{11}$$
$$D_y + DV - \hat{V}D = 0, \tag{12}$$
$$D_t + DW - \hat{W}D = 0, \tag{13}$$

where

$$\hat{U} = \begin{pmatrix} -\frac{1}{2}\lambda & \hat{u} \\ \hat{v} & \frac{1}{2}\lambda \end{pmatrix}, \tag{14}$$

$$\hat{V} = \begin{pmatrix} -\frac{1}{2}\lambda^2 + \hat{u}\hat{v} & \hat{u}\lambda - \hat{u}_x \\ \hat{v}\lambda + \hat{v}_x & \frac{1}{2}\lambda^2 - \hat{u}\hat{v} \end{pmatrix}, \tag{15}$$

$$\hat{W} = \tag{16}$$
$$\begin{pmatrix} -\frac{1}{2}\lambda^3 + \hat{u}\hat{v}\lambda - \hat{v}\hat{u}_x + \hat{u}\hat{v}_x & \hat{u}\lambda^2 - \hat{u}_x\lambda - 2\hat{u}^2\hat{v} + \hat{u}_{xx} \\ \hat{v}\lambda^2 + \hat{v}_x\lambda - 2\hat{u}\hat{v}^2 + \hat{v}_{xx} & \frac{1}{2}\lambda^3 - \hat{u}\hat{v}\lambda + \hat{v}\hat{u}_x - \hat{u}\hat{v}_x \end{pmatrix}.$$

With the aid of symbolic computation, the DTs for (2) and (3) can be obtained in the following.

The first DT

$$D_1 = \begin{pmatrix} \lambda + \frac{u\psi_2(\lambda_1) - \lambda_1\psi_1(\lambda_1)}{\psi_1(\lambda_1)} & -u \\ -\frac{\psi_2(\lambda_1)}{\psi_1(\lambda_1)} & 1 \end{pmatrix} \tag{17}$$

with

$$\hat{u} = \frac{u[u\psi_2(\lambda_1) - \lambda_1\psi_1(\lambda_1)]}{\psi_1(\lambda_1)} - u_x, \quad \hat{v} = \frac{\psi_2(\lambda_1)}{\psi_1(\lambda_1)}, \tag{18}$$

where $(\psi_1(\lambda_1), \psi_2(\lambda_1))^{\mathrm{T}}$ is a solution of Lax representation (5) – (8) with $\lambda = \lambda_1$.

The second DT

$$D_2 = \begin{pmatrix} 1 & -\frac{\psi_1(\lambda_2)}{\psi_2(\lambda_2)} \\ v & \lambda + \frac{-v\psi_1(\lambda_2) - \lambda_2\psi_2(\lambda_2)}{\psi_2(\lambda_2)} \end{pmatrix} \tag{19}$$

with

$$\begin{aligned} \hat{u} &= -\frac{\psi_1(\lambda_2)}{\psi_2(\lambda_2)}, \\ \hat{v} &= v_x + \frac{v[-v\psi_1(\lambda_2) - \lambda_2\psi_2(\lambda_2)]}{\psi_2(\lambda_2)}, \end{aligned} \tag{20}$$

where $(\psi_1(\lambda_2),\psi_2(\lambda_2))^{\mathrm{T}}$ is a solution of Lax representation (5)–(8) with $\lambda=\lambda_2$.

The third DT

$$D_3=\begin{pmatrix} \lambda-\frac{\lambda_4\psi_1(\lambda_3)\psi_2(\lambda_4)-\lambda_3\psi_1(\lambda_4)\psi_2(\lambda_3)}{\psi_1(\lambda_3)\psi_2(\lambda_4)-\psi_1(\lambda_4)\psi_2(\lambda_3)} & -\frac{\lambda_4\psi_1(\lambda_3)\psi_1(\lambda_4)-\lambda_3\psi_1(\lambda_3)\psi_1(\lambda_4)}{\psi_1(\lambda_3)\psi_2(\lambda_4)-\psi_1(\lambda_4)\psi_2(\lambda_3)} \\ -\frac{\lambda_3\psi_2(\lambda_3)\psi_2(\lambda_4)-\lambda_4\psi_2(\lambda_3)\psi_2(\lambda_4)}{\psi_1(\lambda_3)\psi_2(\lambda_4)-\psi_1(\lambda_4)\psi_2(\lambda_3)} & \lambda-\frac{\lambda_3\psi_1(\lambda_4)\psi_2(\lambda_3)-\lambda_4\psi_1(\lambda_3)\psi_2(\lambda_4)}{\psi_1(\lambda_4)\psi_2(\lambda_3)-\psi_1(\lambda_3)\psi_2(\lambda_4)} \end{pmatrix} \tag{21}$$

with

$$\hat{u}=u-\frac{\lambda_4\psi_1(\lambda_3)\psi_1(\lambda_4)-\lambda_3\psi_1(\lambda_3)\psi_1(\lambda_4)}{\psi_1(\lambda_3)\psi_2(\lambda_4)-\psi_1(\lambda_4)\psi_2(\lambda_3)}, \tag{22}$$

$$\hat{v}=v+\frac{\lambda_3\psi_2(\lambda_3)\psi_2(\lambda_4)-\lambda_4\psi_2(\lambda_3)\psi_2(\lambda_4)}{\psi_1(\lambda_3)\psi_2(\lambda_4)-\psi_1(\lambda_4)\psi_2(\lambda_3)}, \tag{23}$$

where $(\psi_1(\lambda_3),\psi_2(\lambda_3))^{\mathrm{T}}$ and $(\psi_1(\lambda_4),\psi_2(\lambda_4))^{\mathrm{T}}$ are two solutions of Lax representation (5)–(8) with $\lambda=\lambda_3$ and $\lambda=\lambda_4$, respectively.

Assume that (u,v) is a compatible solution of (2) and (3), then $(\hat{u},\hat{v})$ defined in the above three DTs are three types of solutions of (2) and (3). Therefore, we know that

$$q=4\hat{u}\hat{v},\quad p=-2\hat{u}^2,\quad r=-2\hat{v}^2 \tag{24}$$

corresponds to a solution of System (1).

3. Multi-Soliton-Like Solutions

In this section, our concern is to construct the multi-soliton-like solutions for System (1) with the above three DTs and (24). Taking $u=-1$, $v=0$ as the seed solutions of (2) and (3), we obtain the basic solution of Lax representation (5)–(8),

$$\begin{aligned}\psi_1(\lambda_j)&=C_1\mathrm{e}^{-\frac{\lambda_j^3}{2}t-\frac{\lambda_j^2}{2}y-\frac{\lambda_j}{2}x}\\&\quad-\lambda_j^{-1}C_2\mathrm{e}^{\frac{\lambda_j^3}{2}t+\frac{\lambda_j^2}{2}y+\frac{\lambda_j}{2}x},\end{aligned} \tag{25}$$

$$\psi_2(\lambda_j)=C_2\mathrm{e}^{\frac{\lambda_j^3}{2}t+\frac{\lambda_j^2}{2}y+\frac{\lambda_j}{2}x},\ (j=1,2,3,4), \tag{26}$$

where C_1, C_2 are arbitrary constants.

3.1. One-Soliton-Like Solutions of System (1)

From the first DT, we can get a solution of (2) and (3),

$$\begin{aligned}u_1=&\Big[\mathrm{e}^{\frac{\lambda_1^3}{2}t+\frac{\lambda_1^2}{2}y+\frac{\lambda_1}{2}x}C_2+\Big(C_1\mathrm{e}^{-\frac{\lambda_1^3}{2}t-\frac{\lambda_1^2}{2}y-\frac{\lambda_1}{2}x}\\&-\lambda_1^{-1}C_2\mathrm{e}^{\frac{\lambda_1^3}{2}t+\frac{\lambda_1^2}{2}y+\frac{\lambda_1}{2}x}\Big)\lambda_1\Big]\Big[C_1\mathrm{e}^{-\frac{1}{2}\lambda_1^3t-\frac{\lambda_1^2}{2}y-\frac{\lambda_1}{2}x}\\&-\lambda_1^{-1}C_2\mathrm{e}^{\frac{\lambda_1^3}{2}t+\frac{\lambda_1^2}{2}y+\frac{\lambda_1}{2}x}\Big]^{-1},\end{aligned} \tag{27}$$

$$v_1=\frac{C_2\mathrm{e}^{\frac{\lambda_1^3}{2}t+\frac{\lambda_1^2}{2}y+\frac{\lambda_1}{2}x}}{C_1\mathrm{e}^{\frac{-\lambda_1^3}{2}t-\frac{\lambda_1^2}{2}y-\frac{\lambda_1}{2}x}-\lambda_1^{-1}C_2\mathrm{e}^{\frac{\lambda_1^3}{2}t+\frac{\lambda_1^2}{2}y+\frac{\lambda_1}{2}x}}. \tag{28}$$

With Transformation (24), a solution of System (1) can be derived as

$$q_1=\frac{4C_1C_2\lambda_1^3\mathrm{e}^{\lambda_1x+\lambda_1^2y+\lambda_1^3t}}{(C_2\mathrm{e}^{\lambda_1x+\lambda_1^2y+\lambda_1^3t}-C_1\lambda_1)^2}, \tag{29}$$

$$p_1=-\frac{2C_1^2\lambda_1^4}{(C_2\mathrm{e}^{\lambda_1x+\lambda_1^2y+\lambda_1^3t}-C_1\lambda_1)^2}, \tag{30}$$

$$r_1=-\frac{2C_2^2\mathrm{e}^{2(\lambda_1x+\lambda_1^2y+\lambda_1^3t)}\lambda_1^2}{(C_2\mathrm{e}^{\lambda_1x+\lambda_1^2y+\lambda_1^3t}-C_1\lambda_1)^2}. \tag{31}$$

3.2. Two-Soliton-Like Solutions of System (1)

Similarly, with the third DT, the two-soliton-like solution of System (1) can be presented,

$$\begin{aligned}q_2=&\,4C_2C_3\delta_1\lambda_3(\lambda_3-\lambda_4)\lambda_4[C_2C_4\delta_2\lambda_4^2-C_1\lambda_3(C_3\delta_3\lambda_3\\&+C_4\lambda_4^2-C_4\lambda_3\lambda_4)][C_1C_3\delta_3\lambda_3\lambda_4+C_2\delta_2(C_3\delta_3\lambda_3\\&-C_3\delta_3\lambda_4-C_4\lambda_3\lambda_4)]^{-2},\end{aligned} \tag{32}$$

$$\begin{aligned}p_2=&-2[C_2C_4\delta_2\lambda_4^2-C_1\lambda_3(C_3\delta_3\lambda_3+C_4\lambda_4^2\\&-\lambda_3C_4\lambda_4)]^2[C_1C_3\delta_3\lambda_3\lambda_4+C_2\delta_2(C_3\delta_3\lambda_3\\&-C_3\delta_3\lambda_4-C_4\lambda_3\lambda_4)]^{-2},\end{aligned} \tag{33}$$

$$\begin{aligned}r_2=&-2C_2^2C_3^2\delta_1^2\lambda_3^2(\lambda_3-\lambda_4)^2\lambda_4^2[C_1C_3\delta_3\lambda_3\lambda_4\\&+C_2\delta_2(C_3\delta_3\lambda_3-C_3\delta_3\lambda_4-C_4\lambda_3\lambda_4)]^{-2},\end{aligned} \tag{34}$$

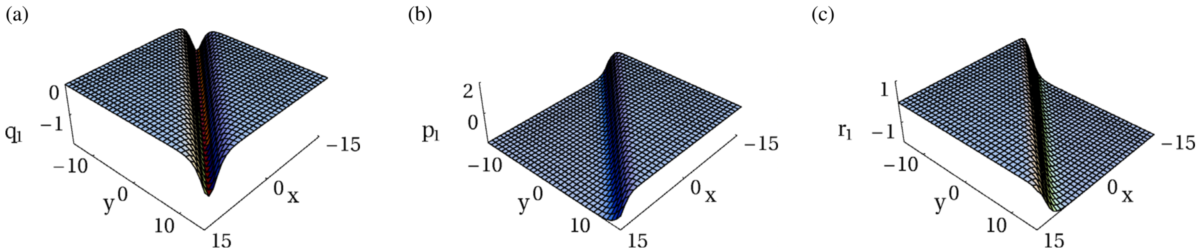

Fig. 1 (colour online). Plots for the three potentials p_1, q_1, and r_1 of the one-soliton-like solution of System (1) transverse at $t=1$. The parameters adopted here are: $c_1=1$, $c_2=1$, $\lambda_1=-1$.

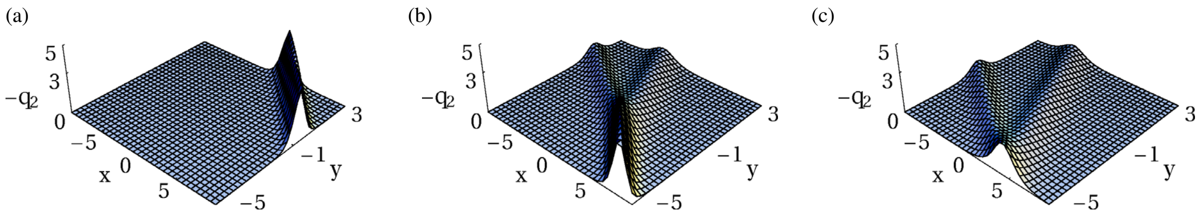

Fig. 2 (colour online). Plots for the potentials q_2 of the two-soliton-like solution of System (1) via Solution (32) transverse at (a) $t=-2$; (b) $t=0$; (c) $t=2$. The parameters adopted here are: $c_1=1$, $c_2=1$, $c_3=1$, $c_4=-1$, $\lambda_3=2$, $\lambda_4=1$.

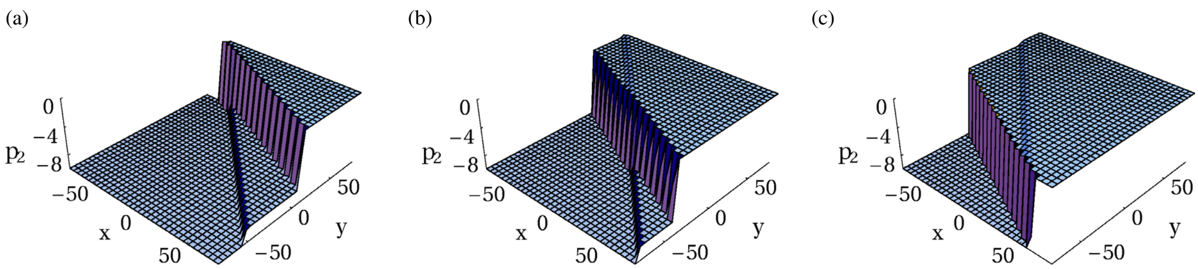

Fig. 3 (colour online). Plots for the potentials p_2 of the two-soliton-like solution of System (1) via Solution (33). The parameters adopted here are: $c_1=1$, $c_2=1$, $c_3=1$, $c_4=-1$, $\lambda_3=2$, $\lambda_4=1$ and depicted at (a) $t=-21$; (b) $t=0$; (c) $t=21$.

where $\lambda_3 \neq \lambda_4$, and

$$\begin{aligned} \delta_1 &= \mathrm{e}^{x(\lambda_3+\lambda_4)+y(\lambda_3^2+\lambda_4^2)+t(\lambda_3^3+\lambda_4^3)}, \\ \delta_2 &= \mathrm{e}^{x\lambda_3+y\lambda_3^2+t\lambda_3^3}, \\ \delta_3 &= \mathrm{e}^{x\lambda_4+y\lambda_4^2+t\lambda_4^3}. \end{aligned} \tag{35}$$

3.3. Three-Soliton-Like Solutions of System (1)

Taking $\lambda_5 \neq \lambda_4 \neq \lambda_3$, with the first basic DT and Transformation (24), we can get the three-soliton-like solution of System (1),

$$\begin{aligned} q_3 = {} & 4C_2\sqrt{\delta_4}\lambda_5(C_2 f_2 \delta_4 \lambda_5^{f_1} + C_1\lambda_5 + C_2\delta_4) \\ & \cdot \left(\frac{C_2 f_2 \delta_4 \lambda_5^{f_2} + C_1}{\sqrt{\delta_4}}\right)^{f_2} \left[C_1\lambda_5 - C_2\delta_4\right]^{-1}, \end{aligned} \tag{36}$$

$$\begin{aligned} p_3 = {} & -\frac{2}{\delta_4}(C_2 f_2 \delta_4 \lambda_5^{f_1} + C_1\lambda_5 + C_2\delta_4)^2 \\ & \cdot \left(\frac{C_2\delta_4 f_2 \lambda_5^{f_2} + C_1}{\sqrt{\delta_4}}\right)^{2f_2}, \end{aligned} \tag{37}$$

$$r_3 = -\frac{2C_2^2\delta_4^2\lambda_5^2}{(C_2\delta_4 - C_1\lambda_5)^2}, \tag{38}$$

where $\delta_4 = \mathrm{e}^{x\lambda_5+y\lambda_5^2+t\lambda_5^3}$, and

$$f_1 = \frac{(C_2\delta_2 - C_1\lambda_3)(\lambda_3-\lambda_4)(C_4\lambda_4 - C_3\delta_3)}{C_2\delta_2(C_4\lambda_3\lambda_4 + C_3\delta_3\lambda_4 - C_3\delta_3\lambda_3) - C_1C_3\delta_3\lambda_3\lambda_4},$$

$$f_2 = \frac{C_2C_4\delta_2\lambda_4^2 - C_1\lambda_3(C_3\delta_3\lambda_3 + C_4\lambda_4^2 - C_4\lambda_4\lambda_3)}{C_1C_3\delta_3\lambda_3\lambda_4 + C_2\delta_2(C_3\delta_3\lambda_3 - C_3\delta_3\lambda_4 - C_4\lambda_3\lambda_4)}.$$

More multi-soliton-like solutions can be obtained by the similar iterative procedure. In the following, Figures 1 – 4 will be drawn via the expressions presented

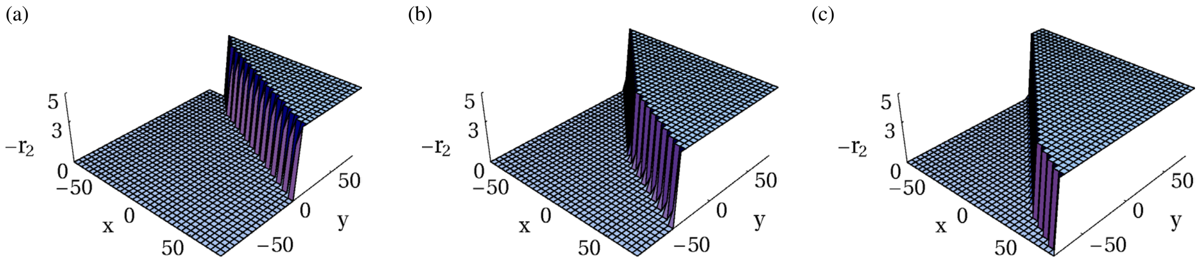

Fig. 4 (colour online). Plots for the potentials r_2 of the two-soliton-like solution of System (1) via Solution (34). The parameters adopted here are: $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, $c_4 = -1$, $\lambda_3 = 2$, $\lambda_4 = 1$ and depicted at (a) $t = -21$; (b) $t = 0$; (c) $t = 21$.

above to depict some dynamics of the obtained solutions.

Figure 1 shows a one-soliton-like solution of System (1), where q_1 has the upside-down bell shape, p_1 and r_1 are the kink-shape solitary waves. Figure 2 displays the resonant structure exhibited by the potential q_2, which describes that the single soliton fissions into two smaller solitons. Solution (32) is determined by six parameters, which means that the interaction states can be organized for different choice of parameters, i. e., one can completely control the phases of the two solitons. Figure 3 illustrates the elastic collision of the two solitons because the soliton amplitudes, velocities, and shapes do not change after their interaction. In Figure 4, the two interacting solitons always keeps a v-shape propagation. In particular, we point out that the soliton resonant phenomenon as shown in Figure 2 has been observed in the shallow water wave experiments [22], in the ion-acoustic waves experiment [23], and in the plasma experiment [24]. Relevant issues can be seen in [26].

4. Conclusions

Recent interest has been found in the algebraic background of a coupled KP system [6]. Some research [5, 6] has claimed that a coupled KP system has an inner connection with matrix integrals over the orthogonal and symplectic ensembles [25], so that the solutions might be applied to the context of random matrix theory [5]. In this paper, with the aid of symbolic computation, we have investigated a coupled KP system with three potentials, i. e., System (1). We have presented three kinds of DTs for the reduced equations of System (1). By using the obtained three DTs and (4), the one-, two-, and three-soliton-like solutions of System (1) have been derived. Those results could be of some value for the studies in the context of random matrix theory [5].

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